

# THERMAL SELF-ENERGIES NEAR ZERO FOUR-MOMENTUM

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## ABSTRACT

We demonstrate that one-loop self-energies at finite temperature have a unique limit as the external four-momentum goes to zero, as long as the particles propagating in the loop have distinct masses. We show that in spontaneously broken theories, this result nonetheless does not affect the difference between screening and propagating modes and hence the usual resummed perturbation expansion remains unaltered.

## 1. Introduction and Motivation

This talk is a review of work that I have done with P. Arnold, P. Bedaque, and A. Das <sup>1</sup>. In finite temperature field theory, the existence of an additional four-vector, namely the four-velocity of the plasma, allows one to construct two independent Lorentz scalars on which all Green's functions, and in particular, polarization tensors and self-energies can depend, namely  $\omega = P \cdot u$  and  $k = ((P \cdot u)^2 - P^2)^{\frac{1}{2}}$ . Here  $u^\mu$  is the four-velocity of the plasma and  $P^\mu = (p^0, \vec{p})$  is the four-momentum of any particle. In the rest-frame of the heat bath, these scalars reduce to  $p^0$  and  $p = |\vec{p}|$  respectively.

This separate dependence allows one to take the limits  $p^0 \rightarrow 0$  and  $p \rightarrow 0$  in different orders. In general, one expects that the limits need not commute, since they correspond to different physical situations. For instance, one may imagine computing the change in the free energy of the QED plasma, after placing two *static* charges  $q_1$  at  $\vec{r}_1$  and  $q_2$  at  $\vec{r}_2$ , as a function of their separation  $r = |\vec{r}_1 - \vec{r}_2|$ . Linear response theory gives the answer

$$U(r) = q_1 q_2 \int \frac{d^3 \vec{k}}{(2\pi)^3} e^{i\vec{k} \cdot \vec{r}} \frac{1}{\vec{k}^2 + \Pi_{00}(0, |\vec{k}|)}. \quad (1)$$

For large separations, the integral is dominated by  $\vec{k} \approx 0$ . So, one may effectively replace

$$\Pi_{00}(0, k) \rightarrow \lim_{k \rightarrow 0} \Pi_{00}(0, k) \equiv \lim_{k \rightarrow 0} \lim_{k^0 \rightarrow 0} \Pi_{00}(k^0, |\vec{k}|). \quad (2)$$

Denoting this double limit by  $m_{el}^2$ , the square of the electric screening mass of the photon, one obtains the usual expression for  $r \rightarrow \infty$ ,

$$U(r) \rightarrow \frac{q_1 q_2}{4\pi} \frac{e^{-m_{el} r}}{r}. \quad (3)$$

My interest in understanding the structure of thermal self-energies near zero four-momentum was motivated by the prospect of baryogenesis at the electroweak phase transition. However, electroweak baryogenesis has brought the need to understand the detailed dynamics of the transition<sup>2</sup>. It is well-known that the validity of perturbation theory at finite temperature is seriously compromised by infrared divergences. Therefore, one needs to resum (infinite) sets of diagrams in order to improve convergence. This means, among other things, using “self-consistent” propagators, which entails replacing tree-level masses by their higher-order values. To achieve that goal, one needs to solve self-consistently the computed dispersion relations

$$P^2 = m^2 + \Pi(P^2), \quad (4)$$

for the location of the physical pole (to that order), and then use that value in an improved propagator. That is where the behavior of thermal self-energies enters with a vengeance. For instance, it has been shown that in the case of hot QCD<sup>3</sup>, self-interacting scalars<sup>4,5</sup>, and gauge theories with chiral fermions<sup>6</sup>, the two aforementioned limits of the self-energy do not indeed commute. Guided by these results, people have been using the non-analytic (in  $p^0$  and  $p$ ) high-temperature expressions in the improved propagators in the Standard Model. In this talk, however, I will demonstrate that there exist contributions to the one-loop self-energy of a massive gauge boson in a spontaneously broken gauge theory, which possess a *unique* limit as  $p$  and  $p^0$  tend to zero, as long as the particles propagating in the loop have different masses. Given that the Standard Model is such a theory, does that invalidate the literature results of carefully computed quantities at the phase transition? The answer is *no*. I will show, that even if one-loop self-energies are perfectly analytic “around zero four-momentum”, the usual approximation which uses the non-commuting limits is the relevant and correct procedure, at least for the purposes of computing *physical* quantities, such as poles of particle propagators.

## 2. Spontaneously Broken U(1) Theory

For simplicity, we will perform the calculation of the polarization tensor for the massive vector boson in the Abelian Higgs model in unitary gauge. Unitary gauge is infamous for complications in the Higgs sector at finite temperature<sup>7</sup>. In the gauge sector, however, these complications are absent and the smaller number of diagrams makes its use preferable for our purposes.

The Lagrangian for the Abelian Higgs model in the unitary gauge is given by

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{e^2v^2}{2}A^\mu A_\mu + \frac{1}{2}\partial^\mu\eta\partial_\mu\eta - \frac{m^2}{2}\eta^2 \\ & + \frac{e^2}{2}A^\mu A_\mu\eta^2 + e^2vA^\mu A_\mu\eta - \lambda v\eta^3 - \frac{\lambda}{4}\eta^4, \end{aligned} \quad (5)$$

where  $\eta$  is the Higgs field,  $A_\mu$  is the U(1) gauge field and the vacuum expectation value,  $v = m/\sqrt{2\lambda}$ . In unitary gauge, there is a single one-loop, momentum-dependent correction to the photon propagator, which we denote by  $\tilde{\Pi}_{\mu\nu}$ . This

diagram gives via the usual methods<sup>8</sup>,

$$\text{Re}\tilde{\Pi}_{00}^\beta = 4e^2 \int \frac{d^3\vec{k}}{(2\pi)^3} \left[ \frac{n(\omega_k)}{2\omega_k} \frac{M^2 - (p_0 + \omega_k)^2}{(p_0 + \omega_k)^2 - \Omega_{k+p}^2} + \frac{n(\Omega_k)}{2\Omega_k} \frac{M^2 - \Omega_k^2}{(p_0 + \Omega_k)^2 - \omega_{k+p}^2} \right] + (p_0 \rightarrow -p_0). \quad (6)$$

Here we have defined  $M = ev$ ,  $\omega_k = \sqrt{\vec{k}^2 + m^2}$  and  $\Omega_k = \sqrt{\vec{k}^2 + M^2}$ . After doing the angular integration, one obtains

$$\text{Re}\tilde{\Pi}_{00}^\beta(p_0, p) = -\frac{e^2}{2\pi^2} \int_0^\infty dk k \left[ \frac{(k^2 + p_0^2 + \Delta)n(\omega_k)}{2\omega_k} \frac{1}{p} \ln |S_1| + \frac{k^2 n(\Omega_k)}{2\Omega_k} \frac{1}{p} \ln |S_2| + n(\omega_k) \frac{p_0}{p} \ln \left| \frac{(p_0^2 - p^2 + \Delta)^2 - 4(p_0\omega_k + pk)^2}{(p_0^2 - p^2 + \Delta)^2 - 4(p_0\omega_k - pk)^2} \right| \right], \quad (7)$$

where  $\Delta = m^2 - M^2$ , and the  $S_i$  are given by the following expression, with  $m_1 = m$  and  $m_2 = M$ ,

$$S_i = \frac{(p_0^2 - p^2 + 2pk - (-1)^i \Delta)^2 - 4p_0^2 \omega_i^2}{(p_0^2 - p^2 - 2pk - (-1)^i \Delta)^2 - 4p_0^2 \omega_i^2}, \quad i = 1, 2. \quad (8)$$

Let us analyze the small- $p^0$ , small- $p$  behavior of Eq. (7). For that purpose, let us set

$$p^0 = \alpha p. \quad (9)$$

Then, for *nonzero* values of  $\Delta = m^2 - M^2$ , it is clear that

$$\lim_{p \rightarrow 0} \text{Re}\tilde{\Pi}_{00}^\beta(\alpha p, p) = -\frac{4e^2}{\pi^2} \int_0^\infty dk \left[ k^2 \frac{n(\omega_k)}{2\omega_k} + \frac{k^4}{m^2 - M^2} \left( \frac{n(\omega_k)}{2\omega_k} - \frac{n(\Omega_k)}{2\Omega_k} \right) \right]. \quad (10)$$

In particular, this limit is finite,  $\alpha$ -independent and hence independent of the ratio  $p_0/p$  as  $p_0$  and  $p$  approach zero. Alternatively, this may be obtained by simply putting  $P^\mu = 0$  in Eq. (6). So, the double limit is unique, as promised. Furthermore, it is easy to establish that  $\text{Re}\tilde{\Pi}_{ii}^\beta$  has a unique double limit as well.

The high-temperature limit of Eq. (10) can be easily obtained to be

$$\lim_{\substack{p \rightarrow 0 \\ p_0 \rightarrow 0}} \text{Re}\tilde{\Pi}_{00}^\beta(p_0, p) = \frac{1}{6} e^2 T^2. \quad (11)$$

This turns out to be the same as the  $(p_0 = 0, \vec{p} \rightarrow 0)$  limit of the equal mass case  $\Delta = 0$  (see also Eq. (13)). Note that even though Eq. (10) appears to be singular when  $m = M$ , it indeed has a finite limit as the two masses become degenerate and corresponds to the  $p_0 = 0, \vec{p} \rightarrow 0$  limit of the degenerate case.

### 3. Debye and Plasmon Masses

At this point it is instructive to compare and contrast the above result with the usual non-commuting double limits. This exercise will shed light on what one means by considering the self-energy “near zero external four-momentum”. Let’s start with Eq. (6). We Taylor expand the denominators of the integrand in the high-temperature limit  $T \gg m_i, p_0, p$ , keeping in mind that  $k \sim T$  (in view of the Bose-Einstein factor). For instance,

$$\begin{aligned} \frac{1}{(p_0 + \omega_k)^2 - \Omega_{k+p}^2} &= \frac{1}{2P \cdot K + (P^2 + m^2 - M^2)} \\ &= \frac{1}{2P \cdot K} \left( 1 - \frac{P^2 + m^2 - M^2}{2P \cdot K} + \dots \right), \end{aligned} \quad (12)$$

where  $K^0$  is on the mass-shell. Then one finds that all masses drop out from the integrand (or can be neglected to leading order). Therefore, it is not surprising to find that the high-temperature limit in *this* regime of external momenta is

$$\tilde{\Pi}_{00}^\beta(p_0, p) = \frac{e^2 T^2}{6} \int \frac{d\Omega_n}{4\pi} \left[ \frac{n_0^2 P^2}{(n \cdot P)^2} - \frac{2n_0 p_0}{n \cdot P} \right], \quad (13)$$

which agrees with the standard Braaten-Pisarski result<sup>3,7</sup>, and which is explicitly non-analytic. Here  $n^\mu = (1, \vec{n})$ , with  $|\vec{n}| = 1$ , and the angular integration is over all possible orientations of that vector.

So, where is the sleight of hand? The same expression, Eq. (6), surely cannot be simultaneously analytic and non-analytic around zero. The answer lies in the study of the validity of the Taylor expansion above. The non-analytic expression was got by assuming that  $P \cdot K \gg \Delta$ , or  $p_0, p \gg |m^2 - M^2|/T$ . For a theory with  $\Delta = 0$ , this is always satisfied. However, with  $\Delta \neq 0$ , there is a region  $p_0, p \ll |m^2 - M^2|/T$ , for which the Taylor expansion above is inappropriate and the analysis of the previous section shows that the self-energy has a unique value around zero.

### 4. Discussion

It is easy to see that the same result holds for a theory with two scalars, as well as for QED with massive fermions: the one-loop self-energy/polarization tensor at finite temperature has a unique limit as the external four-momentum goes to zero. The absence of the usual non-commuting double limits is traced to the fact that there is (generically) a finite mass difference among the particles propagating in the loop. One can understand this result in the following way. The real part of the one-loop self-energy is related to the imaginary part through the dispersion relation<sup>9</sup>,

$$\begin{aligned} \text{Re}\Sigma_R^\beta(p_0, p) &= \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} du \frac{\text{Im}\Sigma_R^\beta(u, p)}{u - p_0} \\ &= \frac{2}{\pi} \mathcal{P} \int_0^{\infty} du u \frac{\text{Im}\Sigma_R^\beta(u, p)}{u^2 - p_0^2}. \end{aligned} \quad (14)$$

The last equality follows from the fact that  $\text{Im}\Sigma_R^\beta(p_0, p)$  is an odd function of  $p_0$ <sup>10</sup>. Here  $\Sigma_R^\beta$  is the retarded two point function related to  $\Sigma^\beta$  by

$$\begin{aligned}\text{Im}\Sigma_R^\beta(u, p) &= \text{Im}\Sigma^\beta(u, p) \tanh \frac{\beta u}{2} \\ \text{Re}\Sigma_R^\beta(u, p) &= \text{Re}\Sigma^\beta(u, p).\end{aligned}\tag{15}$$

As pointed out by Weldon<sup>5</sup>,  $\text{Im}\Sigma_R^\beta(u, p)$  is non-zero only for some values of  $u^2 - p^2$ . The imaginary part of the self-energy is expressed in terms of the discontinuity of  $\Sigma_R^\beta(p_0, p)$  along these cuts on the real axis,

$$\lim_{\epsilon \rightarrow 0^+} \left( \Sigma_R^\beta(p_0 + i\epsilon, p) - \Sigma_R^\beta(p_0 - i\epsilon, p) \right) = -2i \text{Im}\Sigma_R^\beta(p_0, p),\tag{16}$$

for real  $p_0$ . For fixed  $m_1$  and  $m_2$ , these cuts exist for

$$u^2 - p^2 \geq (m_1 + m_2)^2,\tag{17}$$

$$u^2 - p^2 \leq (m_1 - m_2)^2.\tag{18}$$

The first cut is the usual zero-temperature cut corresponding to the decay of the incoming particle, whereas the second appears only at  $T \neq 0$  and represents absorption of a particle from the medium. The first cut does not lend itself to non-commuting double limits, so the only suspect is the second cut. In fact, it is this cut which is responsible for the non-commuting double limits in the case  $m_1 = m_2$ <sup>9</sup>. In our case however, the contribution of this cut is perfectly well-behaved as  $P^\mu \rightarrow 0$ . In fact, if we denote this contribution by  $C_2(p_0, p)$ , then we obtain

$$\text{Re}\Sigma_R^\beta(p_0, p) \ni C_2(p_0, p) = \frac{2}{\pi} \mathcal{P} \int_0^{(p^2 + (m_1 - m_2)^2)^{\frac{1}{2}}} du u \frac{\text{Im}\Sigma_R^\beta(u, p)}{u^2 - p_0^2}.\tag{19}$$

Performing the change of variables  $u \rightarrow u/\sqrt{p^2 + (m_1 - m_2)^2}$ , we obtain

$$\text{Re}\Sigma_R^\beta(p_0, p) \ni C_2(p_0, p) = \frac{2}{\pi} \mathcal{P} \int_0^1 du u \frac{\text{Im}\Sigma_R^\beta(u \sqrt{p^2 + (m_1 - m_2)^2}, p)}{u^2 - \frac{p_0^2}{p^2 + (m_1 - m_2)^2}}.\tag{20}$$

As long as the masses are different, the zero momentum limit of  $C_2(p_0, p)$  is well-defined and given by

$$C_2(0, 0) = \frac{2}{\pi} \int_0^{|m_1 - m_2|} du \frac{\text{Im}\Sigma_R^\beta(u, 0)}{u}.\tag{21}$$

This limit, however, is not well-defined if the masses are equal. Note that Eq. (21) is well-behaved, given that  $\text{Im}\Sigma^\beta(u, 0)$  is odd in  $u$ , and goes as  $u$  for small  $u$ .

One may naturally wonder whether our observation has any effect on standard computations of *physical* quantities, such as the difference between Debye and plasmon masses in the standard electroweak theory, and whether there could be any

effect on studies of the electroweak phase transition. In fact it does not, as one may argue in view of the results of the previous section. There, we noted that our result for the  $P^\mu \rightarrow 0$  limit, Eq. (11), depends on assuming  $p_0, p \ll |\Delta|/T$  in Eq. (6), since Eq. (6) is dominated by  $k \sim T$ . However, the region of interest for self-consistently finding the Debye or plasmon poles of the vector propagator is when  $p_0$  or  $p$  take values of order  $m_i \gg |\Delta|/T$ . In that regime,  $\Delta$  can be ignored in Eq. (7), in which case one recovers the usual non-commuting double limits. For  $p_0$  and  $p$  small compared to  $|\Delta|/T$ , the functions  $\Pi_{00}^\beta(p_0, 0)$  and  $\Pi_{00}^\beta(0, p)$  tend to the same limit. However, at order  $m$ , the functions take on different values. As the mass difference goes to zero, it is clear that the unique limit disappears, as well.

## 5. References

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